Exercise 4 Stochastic Models of Manufacturing Systems 4T400, 19 May

1. A processor can work on two types of tasks. Type-1 tasks arrive according to a Poisson process with a rate of 100 per second and type-2 tasks according to a Poisson process with a rate of 200 per second. The two arrival processes are independent. Both types of tasks have exponentially distributed service times, with a mean of 3 milliseconds. Tasks are processed in order of arrival.

   (a) What is the probability that during 10 milliseconds no new tasks arrive?

   (b) What is the probability that the first next task to arrive is of type-1?

   (c) Determine the limiting probability of n tasks at the processor, $n = 0, 1, 2, \ldots$.

   (d) Determine the mean number of tasks at the processor.

   (e) Determine the mean sojourn time (waiting time plus processing time) of a task.

   Answer:

   (a) Let λ_1 be the arrival rate of type-1 tasks, and λ_2 the arrival rate of type-2 tasks. So $\lambda_1 = 0.1$ and $\lambda_2 = 0.2$ tasks per millisecond. The arrivals of type-1 and type-2 tasks are also Poisson with rate $\lambda_1 + \lambda_2 = 0.3$ tasks per millisecond. Hence, the probability that during 10 milliseconds no new task arrives is

   $$P(\text{no arrival in } [0, 10]) = e^{-(\lambda_1+\lambda_2)\cdot 10} = e^{-3} \approx 0.05$$

   (b) The probability that the first next task is of type-1 is $\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{1}{3}$.

   (c) Let ρ be the utilization of the processor, so $\rho = (\lambda_1+\lambda_2)\cdot 3 = 0.9$. Hence, the probability of n tasks at the processor is

   $$p_n = 0.1 \cdot 0.9^n, \quad n = 0, 1, 2, \ldots$$

   (d) The mean number $E(L)$ is

   $$E(L) = \frac{\rho}{1-\rho} = 9.$$

   (e) By Little's law, the mean sojourn time $E(S)$ is given by

   $$E(S) = \frac{E(L)}{\lambda_1 + \lambda_2} = 30 \quad \text{(milliseconds)}.$$

2. In a helpdesk, two operators are servicing incoming calls. The service times are exponential with mean 30 seconds. Calls arrive according to a Poisson stream with rate 2 calls per minute. Incoming calls are handled in order of arrival. Let $X(t)$ denote the number of calls in the system (waiting and in service) at time t.

   (a) What is the probability that at least 3 calls have arrived in $[0, t]$?

   (b) Specify the flow diagram of the process $X(t)$.

   (c) Determine the limiting probabilities p_n of n calls in the system, $n = 0, 1, 2, \ldots$.

   (d) Determine the probability that an incoming call has to wait before being handled by one of the operators.

   (e) Determine the mean sojourn time (waiting time plus service time) of a call.

   Answer:

(a) Calls arrive according to a Poisson stream with rate 2 (calls/min). Hence, the probability of at least 3 arrivals is $1 - e^{-2t}(1 + 2t + 2t^2)$.

(b) This is an $M/M/2$ system, with arrival and service rate equal to 2 (per min). Let $q_{n,m}$ denote the rate from state n to m, then the nonzero rates are $q_{n,n+1} = 2$ for $n \geq 0$, $q_{n,n-1} = 4$ for $n \geq 2$ and $q_{1,0} = 2$.

(c) Let p_n be the limiting probability of n calls in the system, then $p_n = \frac{1}{3}\left(\frac{1}{2}\right)^{n-1}$ for $n \geq 1$ and $p_0 = \frac{1}{3}$.

(d) $P_{wait} = p_2 + p_3 + \cdots = 1 - p_0 - p_1 = \frac{1}{3}$.

(e) Let L denote the number is the system and S the sojourn time. Then

$$E(L) = \sum_{n=1}^{\infty} np_n = \frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{4}{3}.$$

So by Little's law,

$$E(S) = \frac{E(L)}{2} = \frac{2}{3} \text{ (min)}.$$

Or: Let W denote the waiting time and Q the number in the queue. Then by PASTA and Little,

$$\begin{aligned} E(W) &= P_{wait} \cdot \frac{1}{4} + E(Q) \cdot \frac{1}{4} \\ E(Q) &= 2E(W), \end{aligned}$$

so $E(W) = \frac{1}{6}$ (min) and thus $E(S) = E(W) + \frac{1}{2} = \frac{2}{3}$ (min).

3. Consider a machine where jobs arrive according to a Poisson stream with a rate of 4 jobs per hour. Half of the jobs has a processing time of exactly 10 minutes, a quarter of the jobs has a processing time of exactly 15 minutes and the remaining quarter has a processing time of 20 minutes. The jobs with a processing time of 10 minutes are called type 1 jobs, the ones with a processing time of 15 minutes type 2 jobs and the rest type 3 jobs. The jobs are processed in order of arrival. Determine the mean sojourn time (waiting time plus processing time) of a type 1, 2 and 3 job and also of an arbitrary job.

Answer: Let B denote the processing time of an arbitrary job, and let R denote the residual processing time. Then

$$E(B) = \frac{1}{2} \cdot 10 + \frac{1}{4} \cdot 15 + \frac{1}{4} \cdot 20 = 13\frac{3}{4}\text{(min)}, E(B^2) = \frac{1}{2} \cdot 10^2 + \frac{1}{4} \cdot 15^2 + \frac{1}{4} \cdot 20^2 = 206\frac{1}{4},$$

so the utilization $\rho = \frac{1}{15} \cdot E(B) = \frac{11}{12}$ and

$$E(R) = \frac{E(B^2)}{2E(B)} = 7.5 \text{ (min)}.$$

So the mean waiting time is

$$E(W) = \frac{\rho E(R)}{1 - \rho} = 82.5 \text{ (min)},$$

and for the sojorun times we get

$$E(S_1) = E(W) + 10 = 92.5 \text{ (min)}, \quad E(S_2) = 97.5 \text{ (min)}, \quad E(S_3) = 102.5 \text{ (min)}$$

and

$$E(S) = E(W) + E(B) = 96.25 \text{ (min)}.$$

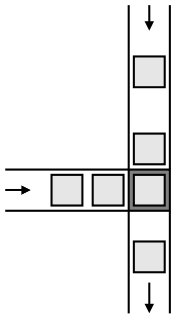

Figure 1: Junction of an automatic conveyor system transporting pallets.

4. In a distribution center pallets with products are transported on an automatic conveyor system. In Figure 1 a junction is shown, where pallets from the West and North join the main conveyor belt. The time to pass the transfer point (dark square in Figure 1) is exactly 8 seconds for a pallet coming from the North. Pallets from the West first need to be lifted a little bit, and therefore the time to pass the transfer point is longer, i.e., it is exactly 12 seconds for pallets from the West. Pallets arrive at the transfer point according to a Poisson process, with a rate of 3 pallets per minute from the North and 2 pallets per minute from the West. Calculate the mean time to pass the transfer point (waiting time plus transfer time) for a pallet from the North and for a pallet from the West in case pallets are transferred in order of arrival.

   Answer: As time unit we choose 1 second:

$$\lambda_N = 1/20,\ E(B_N) = 8,\ \rho_N = 2/5,\ E(R_N) = 4,$$

$$\lambda_W = 1/30,\ E(B_W) = 12,\ \rho_W = 2/5,\ E(R_W) = 6,$$

$$\lambda = 1/12,\ E(B) = 3/5 \cdot 8 + 2/5 \cdot 12 = 48/5,\ \rho = 4/5,\ E(R) = 1/2 \cdot 4 + 1/2 \cdot 6 = 5.$$

   So $E(W) = \frac{\rho}{1-\rho} E(R) = 20$ seconds. Hence, $E(S_N) = 28$ and $E(S_W) = 32$ seconds.

5. At a machine jobs arrive according to a Poisson process with a rate of 24 jobs per hour. With probability $(1/2)^i$ the process time of a job is i minutes, $i = 1, 2, \ldots$. Jobs are processed in order of arrival.

   (a) Determine the mean and variance of the process time of a job.

   (b) Determine the mean waiting time of an arbitrary job.

   (c) Determine the mean flow time (waiting time plus process time) of an arbitrary job.

   (d) Determine the mean flow time of a job, the process time of which is 2 minutes.

   Answer:

   (a)

$$E(B) = \sum_{i=1}^{\infty} i(1/2)^i = 2,$$

   and

$$E(B^2) = \sum_{i=1}^{\infty} i^2(1/2)^i = \sum_{i=1}^{\infty} i(i-1)(1/2)^i + \sum_{i=1}^{\infty} i(1/2)^i = 4 + 2 = 6.$$

   So

$$\mathrm{var}(X) = E(B^2) - (E(B))^2 = 6 - 4 = 2 \text{ (min)}.$$

(b) The mean residual process time is

$$E(R) = \frac{E(B^2)}{2E(B)} = \frac{3}{2} \text{ (min)}.$$

Since $\rho = \lambda E(B) = \frac{2}{5} \cdot 2 = \frac{4}{5}$, we get for the mean waiting time,

$$E(W) = \frac{\rho E(R)}{1-\rho} = 6 \text{ (min)}.$$

(c) $E(S) = E(W) + E(B) = 8$ (min).

(d) The mean flow time is $E(W) + 2 = 8$ (min).